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SUPERCRITICAL SPEED STABILITY OF THE TRIVIAL EQUILIBRIUM OF AN AXIALLY-MOVING STRING ON AN ELASTIC FOUNDATION†

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The stability of an axially-moving string supported by a discrete or distributed elastic foundation is examined analytically. Particular attention is directed at the distribution of the critical speeds and identifying the divergence instability of the trivial equilibrium. The elastically supported string shows unique stability behavior that is considerably different from unsupported axially-moving string. In particular, any elastic foundation (discrete or distributed) leads to multiple critical speeds and a single region of divergence instability above the first critical speed, whereas the unsupported string has one critical speed and is stable at all supercritical speeds. Additionally, the elastically supported string critical speeds are bounded above, and the maximum critical speed is the upper bound of the divergent speed region. The analysis draws on the self-adjoint eigenvalue problem for the critical speeds and a perturbation analysis about the critical speeds. Neither numerical solution nor spatial discretization, which can produce substantially incorrect results, are required. The system falls in the class of dispersive gyroscopic continua, and its behavior provides a useful comparison for general gyroscopic systems. The analytical findings also serve as a benchmark for approximate methods applied to gyroscopic continua.

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1. INTRODUCTION

Axially moving strings have been the subject of much research because of their use in modelling the transverse vibration of a variety of physical systems, e.g., tape drives, power transmission belts, paper handling, textile manufacturing, and filament processing. Prior research has provided a thorough understanding of the natural frequencies, eigenfunctions, wave propagation, stability, and response of the moving string supported only at its endpoints. Comprehensive literature reviews are given in references [2, 3]. This work addresses axially moving strings supported by elastic foundations. This model can be used to represent moving

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media supported by air bearings, recording heads, tensioner arms, and the like. Moving media systems have inherent theoretical appeal as prototypical gyroscopic continua. The elastically supported moving string is of particular interest as the elastic foundation makes the system dispersive. As such, it provides a useful comparison for other important dispersive, gyroscopic continua such as spinning disks, spinning shafts, pipes transporting fluids, and translating beams. The elastic foundation creates unique stability phenomena that contrast sharply with the unsupported moving string behavior and have not been observed previously in gyroscopic continua. These phenomena are the focus of this study.

This work examines the supercritical speed dynamics with particular focus on the distribution of the critical speeds and stability of the trivial equilibrium. Critical speeds are defined as those speeds where the system has a vanishing eigenvalue and is subject to a buckling instability. Transport speeds above the lowest critical speed are termed supercritical. While an elastic foundation does not alter the lowest critical speed [4], the supercritical speed stability picture is dramatically changed by the elastic foundation. The importance of modelling geometric non-linearities in free and forced response analyses increases with translation speed [5, 6]. These non-linearities admit multiple equilibria that bifurcate from the trivial one [7]. This study addresses the local stability of the trivial equilibrium. The cases of n discrete springs, a partial elastic foundation over a segment of the span, and a complete elastic foundation are examined. In each case, well-defined regions of divergence instability caused by the elastic foundations are identified. Identification of these unstable regions does not require spatial discretization or numerical approximation. Stability conclusions are determined entirely from the solvable critical speed eigenvalue problem and a perturbation analysis about the critical speeds. Avoidance of discretization is important as such methods can yield substantially erroneous qualitative and quantitative results for this problem. The analytical results provide a benchmark for evaluating approximate methods applied to gyroscopic systems.

Previous works on elastically supported, moving strings are limited. Bhat *et al*. [8] used a finite difference discretization to investigate the response of elastically supported travelling strings. Perkins [4] examined the subcritical eigensolutions for a string supported by a discrete spring, partial foundation, and complete foundation. He also points out the significant effect the elastic foundation can have on the response. Tan and Zhang [9] examine the natural frequencies, mode localization, and response of translating strings on distributed elastic foundations. Chen [10] calculated the natural frequencies for a discrete mass–spring–damper support using a Galerkin discretization. Wickert [11] presents a Green's function solution for arbitrary transverse excitation. None of these works consider supercritical speeds.

2. AXIALLY MOVING STRING ON AN ELASTIC FOUNDATION

Figure 1 depicts the axially moving string under consideration. The string is under constant tension P and translates with constant speed V. The mass per unit length is ρ. The discrete elastic supports are of stiffness κ_i located at D_i, while the

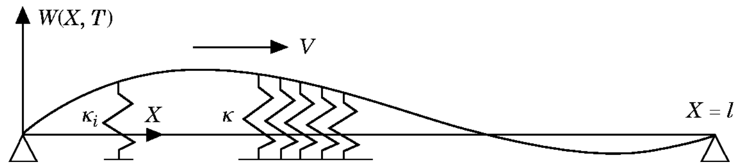

Figure 1. Axially translating string supported by discrete springs of stiffness κ_i and a partial elastic foundation of stiffness κ.

distributed foundation is of stiffness per unit length κ extending over the interval $D_l \leqslant X \leqslant D_r$. Linearization of the geometrically non-linear equations of motion [6] about the trivial equilibrium yields the governing equation for small transverse deflection,

$$w_{tt} + 2vw_{xt} - (1 - v^2)w_{xx} + \sum_{j=1}^{N} k_i \delta(x - d_i)w + k[\mathrm{H}(x - d_l) - \mathrm{H}(x - d_r)]w = 0, \quad (1)$$

where the following dimensionless variables are used:

$$x = X/l, \qquad w = W/l, \qquad t = \sqrt{P/\rho l^2}\,T, \qquad v = \sqrt{\rho/P}\,V, \qquad d_i = D_i/l,$$
$$k_i = \kappa_i l/P, \qquad k = \kappa l^2/P. \quad (2)$$

The ends of the string are fixed. Equation (1) is expressed as

$$Mw_{tt} + vGw_t + (K + K_d + K_c - v^2\tilde{K})w = 0 \quad (3)$$

with use of the differential operators

$$M = I, \qquad G = 2\,\partial/\partial x, \qquad K = -\partial^2/\partial x^2, \qquad K_d = \sum_{j=1}^{N} k_i \delta(x - d_i),$$
$$K_c = k[\mathrm{H}(x - d_l) - \mathrm{H}(x - d_r)], \qquad \tilde{K} = -\partial^2/\partial x^2. \quad (4)$$

The inertia (M) and stiffness (K, K_d, K_c, $\tilde{K}$) operators are self-adjoint with respect to the inner product $(f, g) = \int_0^1 f\bar{g}\,\mathrm{d}x$; the gyroscopic operator G is skew–self-adjoint. Furthermore, M, K, $\tilde{K}$ are positive definite and K_d, K_c are positive semidefinite. The eigenvalue problem is obtained from the separable solution $w(x, t) = u(x)\,\mathrm{e}^{\lambda t}$:

$$\lambda^2 Mu + \lambda vGu + (K + K_d + K_c - v^2\tilde{K})u = 0, \qquad u(0) = u(1) = 0. \quad (5)$$

3. ANALYSIS OF CRITICAL SPEEDS

Gyroscopic systems such as the moving string are subject to buckling-type instabilities at the critical speeds where one or more system eigenvalues vanish. Critical speeds also frequently form the boundary between stable and unstable operating speeds [12–14]. The critical speeds and associated critical speed eigenfunctions result from the self-adjoint problem

$$\bar{K}u = (K + K_d + K_c - v^2\tilde{K})u = 0, \qquad u(0) = u(1) = 0. \quad (6)$$

The lowest critical speed occurs when $\bar{K}$ ceases to be positive-definite. At supercritical speeds, gyroscopic systems such as the axially moving string *may* be stable despite the stiffness operator being sign indefinite or even negative definite. The inner product

$$(\bar{K}\psi, \psi) = \int_0^1 (1 - v^2)\left(\frac{d\psi}{dx}\right)^2 dx + \sum_{i=1}^{N} k_i \psi^2(d_i) + \int_{d_l}^{d_r} k\psi^2 \, dx \tag{7}$$

reveals that $\bar{K}$ is positive definite for $v^2 < 1$; no critical speeds exist in this range. For $v^2 = 1$, $\bar{K}$ is positive semi-definite (except in the case of a complete elastic foundation, in which case $\bar{K}$ is positive definite). For $v^2 > 1$, $\bar{K}$ is negative definite in the absence of any elastic foundation, and sign indefinite for any combination of discrete or partial foundation. For $v^2 > 1$ and a complete, uniform elastic foundation, an inequality from Dym [15] reduces equation (7) to

$$(\bar{K}\psi, \psi) \leqslant \int_0^1 [k - \pi^2(v^2 - 1)]\psi^2 \, dx. \tag{8}$$

In this case, $\bar{K}$ is negative definite for $v^2 > 1 + k/\pi^2$. A previous paper mistakenly concluded positive definiteness for $v^2 < 1 + k/\pi^2$ and that the lowest critical speed was therefore $v^2 = 1 + k/\pi^2$. In fact, $v^2 = 1 + k/\pi^2$ will be shown to be the *maximum* critical speed (and the maximum unstable speed) for a system with complete elastic foundations.

3.1. NO ELASTIC SUPPORT

In the absence of elastic support ($K_d = K_c = 0$), the system stiffness operator vanishes for $v^2 = 1$. Thus, $v^2 = 1$ is a critical speed of infinite multiplicity (i.e., an infinite number of vanishing eigenvalues), and any continuous function satisfying the boundary conditions is a critical speed eigenfunction. The exact eigenvalues of equation (5) are $\lambda_n = \pm in\pi(1 - v^2)$, $n = 1, 2, \ldots,$ confirming that all eigenvalues vanish at $v^2 = 1$. For a string with no elastic supports, this is the only critical speed and in fact the only unstable speed (i.e., no flutter instability).

3.2. DISCRETE ELASTIC SUPPORTS

The critical speed spectrum for arbitrary discrete spring supports and an analogy to a familiar discrete system vibration problem are found by considering n arbitrarily spaced, discrete springs. The critical speed eigenvalue problem is

$$-(1 - v^2)u_{xx} = 0, \qquad x \neq d_i, \qquad u(d_i^-) = u(d_i^+),$$

$$-(1 - v^2)u_x\big|_{d_i^-}^{d_i^+} + k_i u(d_i) = 0, \qquad i = 1, 2, \ldots, n; \qquad u(0) = u(1) = 0. \tag{9}$$

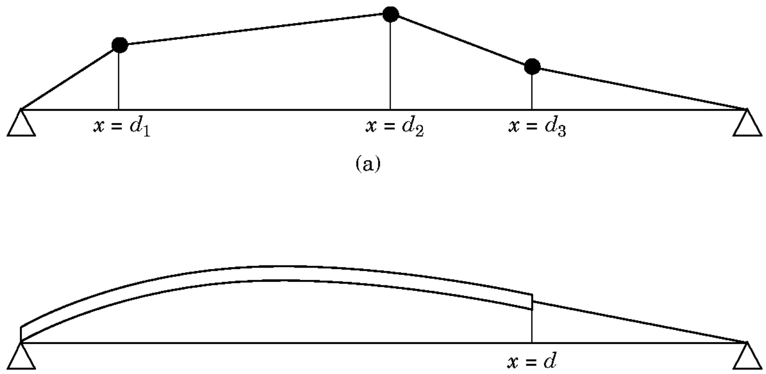

Figure 2. (a) Taut, stationary string of negligible inertia supporting point masses. (b) Taut, stationary string with inertia ρ along a continuous segment $0 < x < d$ and negligible inertia along the remaining portion.

By inspection, $v^2 = 1$ is a critical speed of infinite multiplicity with the associated eigenfunctions being all comparison functions satisfying $u(d_i) = 0$, $i = 1, 2, \ldots, n$. For $v^2 \neq 1$, equation (9) is expressed as

$$Ku - \xi^2 K_d u = -u_{xx} - \xi^2 \sum_{i=1}^{n} k_i \delta(x - d_i) u = 0, \qquad u(0) = u(1) = 0, \tag{10}$$

where $\xi^2 = 1/(v^2 - 1)$. For given numerical values equation (9), and equivalently equation (10), is solvable by matching piecewise-linear string deflections in each segment using the matching conditions at $x = d_i$ in equation (9). It is possible, however, to gain additional insight by relating this problem to the familiar eigenvalue problem for free vibration of a light, stationary, taut string with attached point masses as shown in Figure 2(a). For such a system with tension P and point masses M_i, the dimensionless free vibration eigenvalue problem is

$$-u_{xx} - \omega^2 \sum_{i=1}^{n} m_i \delta(x - d_i) u = 0, \tag{11}$$

where $m_i = M_i/(\Sigma M_j)$, $\tau = l(\Sigma M_j)/P$, ω^2 is the dimensionless natural frequency, and other quantities are as in equation (2). The operators in equation (11) are exactly those in equation (10). Thus, the critical speed eigensolutions of the elastically supported, moving string are calculable from the free vibration eigenvalue problem for a stationary taut string with attached masses with the following associations: (1) the masses of the point inertias are identified with the spring stiffnesses, and (2) the natural frequencies ω^2 of the point mass problem are identified with ξ^2. The taut string with attached masses problem is a discrete system readily formulated in terms of matrix operators. Consequently, one finds the surprising result that the critical speeds of a discretely supported, moving string (a continuous system) are found from the following algebraic (rather than differential) eigenvalue problem:

$$\mathbf{Ku} = \xi^2 \mathbf{Mu}, \qquad \mathbf{u} = (u(d_1)\ u(d_2) \cdots u(d_n))^{\mathrm{T}},$$

$$M_{ii} = k_i, \qquad K_{ii} = 1/(d_i - d_{i-1}) + 1/(d_{i+1} - d_i),$$

$$K_{i(i-1)} = K_{(i-1)i} = -1/(d_i - d_{i-1}), \qquad K_{i(i+1)} = K_{(i+1)i} = -1/(d_{i+1} - d_i). \quad (12)$$

M is diagonal and **K** is symmetric and tri-diagonal. Both **M** and **K** are positive definite, ensuring $\xi^2 > 0$. Thus, n discrete springs generate exactly n additional distinct critical speeds $v_{j+1}^2 = 1 + 1/\xi_j^2$, $j = 1, 2, \ldots, n$, where the index is shifted to account for $v_1^2 = 1$. These are all greater than $v^2 = 1$. At each of the v_{j+1}^2 there exists a single, piecewise-linear critical speed eigenfunction given by the corresponding vibration mode of the stationary string–point mass system. Note that the lowest (highest) critical speed above unity is associated with the highest (lowest) natural frequency of the point mass system. This is reflected in the critical speed eigenfunctions of Figure 3, where one observes that the lower critical speeds are counter-intuitively associated with more deformed eigenfunctions.

Stability conclusions near the critical speeds are available from the following result [12]: for gyroscopic systems of the form (3) having critical speeds with one associated eigenfunction, an eigenvalue always changes from stable (imaginary) to unstable (real) or *vice versa* across a critical speed provided $\tilde{K}$ is positive definite. This is the case here. In general, the eigenvalue derivative at a critical speed is [12]

$$\mu = \partial(\lambda^2)/\partial(v^2)|_{v^2 = v_{\mathrm{crit}}^2} = (\tilde{K}u_{\mathrm{crit}}, u_{\mathrm{crit}})/[(Mu_{\mathrm{crit}}, u_{\mathrm{crit}}) - (Gu_{\mathrm{crit}}, \psi)], \quad (13)$$

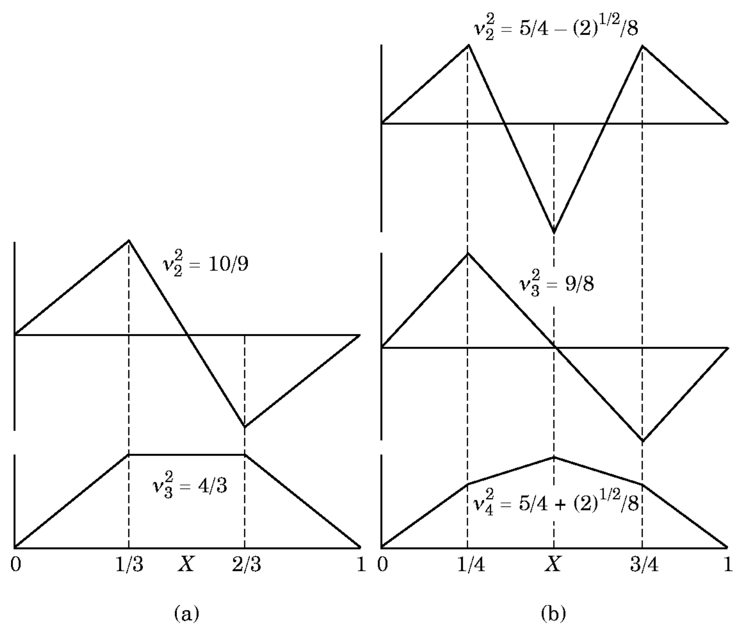

Figure 3. Critical speeds and critical speed eigenfunctions for equally-spaced, discrete spring supports with stiffness $k_i = 1$: (a) two spring supports, and (b) three spring supports. In each case, $v_1^2 = 1$.

where v_{crit} is the critical speed eigenfunction and ψ satisfies

$$(K + K_d - v_{\text{crit}}^2\tilde{K})\psi = -v_{\text{crit}}^2 G u_{\text{crit}}; \tag{14}$$

$\mu \neq 0$ for positive definite $\tilde{K}$, thus ensuring a stability change occurs at each v_{j+1}^2. In contrast to the no elastic support case where all eigenvalues are stable (imaginary) for $v^2 > 1$, each discrete spring: (1) alters one eigenvalue such that it is unstable (real) in a speed range above $v^2 = 1$ while other eigenvalue loci are imaginary above $v^2 = 1$, and (2) introduces one additional critical speed at which the unstable eigenvalue passes back to stability. The key qualitative difference is that a *single*, *bounded* speed range of divergence instability *always* exists above $v^2 = 1$ for one or more discrete springs whereas $v^2 = 1$ is the only unstable speed for no elastic supports. There are a finite number of critical speeds, and the trivial equilibrium is divergent unstable if, and only if, $1 < v^2 < v_{\max}^2$. Flutter is the only possible instability mechanism for $v^2 > v_{\max}^2$ (through the occurrence of flutter is not established).

As an example, consider a string supported along its span by a single discrete spring. $v^2 = 1$ is a critical speed of infinite multiplicity as it is for the unsupported string; the associated critical speed eigenfunctions are all comparison functions satisfying $u(d_1) = 0$. For $v^2 \neq 1$, solution of equation (9) yields the single, additional critical speed eigensolution

$$v_{\text{crit}}^2 = 1 + k_1 d_1(1 - d_1) > 1, \qquad u_{\text{crit}} = \begin{Bmatrix} x, & 0 < x < d_1, \\ [d_1/(1-d_1)](1-x), & d_1 < x < 1, \end{Bmatrix} \tag{15}$$

and the discrete spring introduces one additional critical speed. As $k_1 \to 0$ or $d_1 \to 0, 1$ this critical speed approaches $v^2 = 1$ from above and the unsupported string case is recovered. This second critical speed is maximized for a central spring and decreases symmetrically as the spring is moved to the left or right of center. Stability behavior is predicted by equations (13) and (14), which give

$$\psi = \begin{Bmatrix} -(v_{\text{crit}}^2/k_1 d_1(1-d_1))x^2, & 0 < x < d_1, \\ -(v_{\text{crit}}^2/k_1(1-d_1)^2)x(1-x), & d_1 < x < 1, \end{Bmatrix}, \qquad \mu = -3k_1; \tag{16}$$

$\mu < 0$ ensures that the critical eigenvalue changes from unstable (real) to stable (imaginary) at the critical speed. Notice that μ is independent of spring location, although v_{crit}^2 is not.

The eigenvalue problem (5) for arbitrary $v \neq 1$ has the general solution [4]

$$u = \begin{Bmatrix} A\, e^{[\lambda v/(v^2-1)]x}[e^{-[\lambda/(v^2-1)]x} - e^{[\lambda/(v^2-1)]x}], & 0 < x < d_1, \\ B\, e^{-[\lambda v/(v^2-1)]x}[e^{-[\lambda/(v^2-1)]x} - e^{-2\lambda/(v^2-1)}\, e^{[\lambda/(v^2-1)]x}], & d_1 < x < 1. \end{Bmatrix} \tag{17}$$

Conditions at $x = d_1$ yield the characteristic equation

$$2\lambda[1 - e^{-2\lambda/(v^2-1)}] - k_1[1 + e^{-2\lambda/(v^2-1)} - e^{-2\lambda d_1/(v^2-1)} - e^{-2\lambda(1-d_1)/(v^2-1)}] = 0. \tag{18}$$

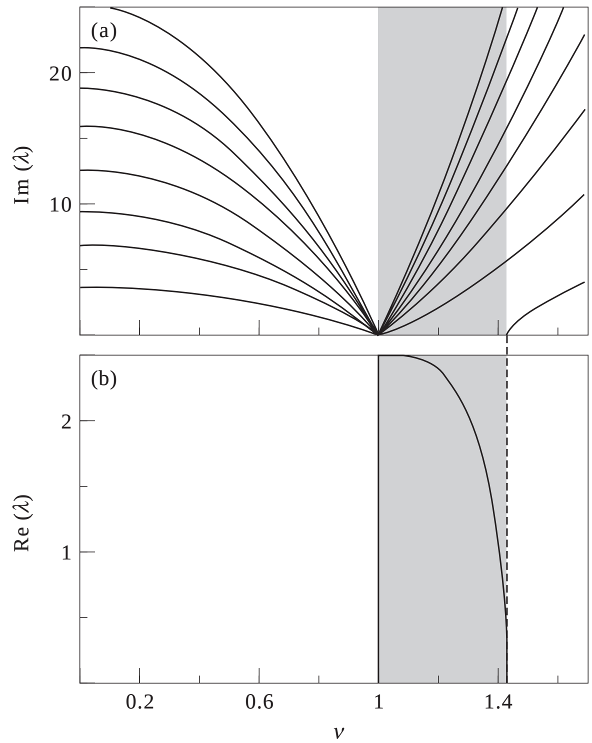

Figure 4. (a) Imaginary and (b) real parts of the eigenvalues of an axially-moving string supported by a discrete spring with $k_1 = 5$ and $d_1 = 0{\cdot}3$. The shaded region indicates divergence instability; no divergence instability exists for $v < 1$ and $v > 1{\cdot}432$.

Eigenvalues λ are shown in Figure 4 for $k_1 = 5$, $d_1 = 0{\cdot}3$. As predicted above, one eigenvalue becomes unstable at $v_1^2 = 1$, and this eigenvalue becomes stable again at the only other critical speed $v_2^2 = 1 + k_1 d_1(1 - d_1)$. The behavior near $v^2 = 1$ is unusual as the real part of the unstable eigenvalue is discontinuous at this speed. The real part jumps from zero to $k_1/2$ across $v^2 = 1$, where the upper value is determined from the limit of equation (18) as $v^2 \to 1^+$ for positive λ. The author is unaware of similar behavior in physical gyroscopic systems. The cause is likely the dramatic change in the stiffness operator at $v^2 = 1$, where the highest order derivative term vanishes. Singular perturbation analysis may clarify the eigenvalue behavior near $v^2 = 1$.

The danger of spatial discretization applied to this problem is evident in the results of Figure 5, which shows the eigenvalues as determined by a twenty term Galerkin discretization of equation (5). The expansion functions consist of the stationary string eigenfunctions. While the low speed eigenvalues and divergence boundaries (solid lines) are predicted, obvious errors are present. These include the predicted flutter instabilities (dashed lines), the failure to capture the eigenvalue discontinuity at $v = 1$, and the imaginary part errors near $v = 1$. These errors persist (and the spurious flutter predictions actually multiply) when forty terms are used in the Galerkin expansion. Use of the complex, speed-dependent eigenfunctions of an axially moving string without an elastic foundation in the Galerkin expansion may improve the results [16].

3.3. PARTIAL ELASTIC FOUNDATION

For a string supported on a distributed elastic foundation, the critical speed eigenvalue problem is

$$-u_{xx} - [k/(v^2 - 1)][1 - H(x - d)]u = 0, \qquad u(0) = u(1) = 0, \tag{19}$$

where the foundation is uniform and extends over the interval $0 < x \leqslant d$. $v^2 = 1$ is again a critical speed of infinite multiplicity, and any comparison function having $u(x) = 0$ for $0 < x \leqslant d$ and $u_x(d^+) = u_x(d^-) = 0$ is a critical speed eigenfunction. For $v^2 > 1$, one compares the critical speed eigenvalue problem to the one for the natural frequencies ω of a stationary, taut string with mass per unit length ρ for $0 < x \leqslant d$ and massless for $d < x \leqslant 1$ (Figure 2(b)) and dimensionless parameters as in equation (2):

$$-u_{xx} - \omega^2[1 - \mathrm{H}(x - d)]u = 0, \qquad u(0) = u(1) = 0. \tag{20}$$

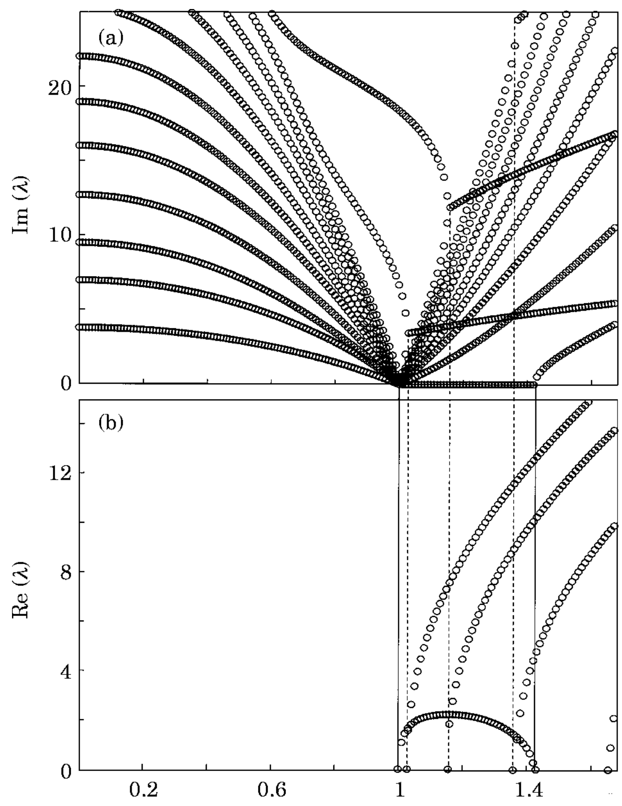

Figure 5. (a) Imaginary and (b) real parts of the eigenvalues of an axially-moving string supported by a discrete spring ($k_1 = 5$, $d_1 = 0{\cdot}3$) using a Galerkin discretization and 20 stationary string eigenfunctions in the expansion. Divergence boundaries are marked by solid lines and flutter boundaries are marked by dashed lines.

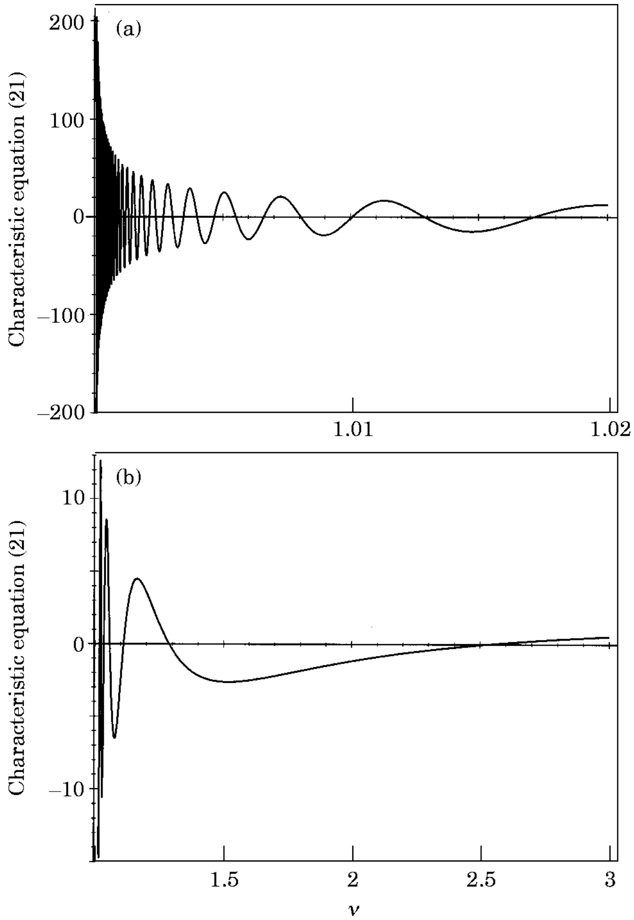

Figure 6. Critical speed characteristic equation (21) for a partial elastic foundation with $k = 20$ and $d = 0{\cdot}6$. (a) Magnified view near $v^2 = 1$, and (b) expanded range. No zeros, and thus no divergence instabilities exist above $v^2 = 2{\cdot}523$.

The critical speeds are related to the analogous stationary string natural frequencies by $v_{j+1}^2 = 1 + k/\omega_j^2$, where the index allows for the critical speed $v_1^2 = 1$. There are an infinite number of critical speeds above unity. Note, however, that these critical speeds are bounded such that $1 \leqslant v^2 \leqslant 1 + k/\omega_1^2$. This is unusual in gyroscopic continua where the spectrum of critical speeds typically approaches infinity. Because $\lim_{j\to\infty} \omega_j^2 = \infty$, critical speeds exist arbitrarily close to $v^2 = 1$. Again, the lowest (highest) critical speed above unity is associated with the highest (lowest) natural frequency of the stationary string system. The critical speed characteristic equation and critical speed eigenfunctions are

$$\sin \xi_{\text{crit}} d + (1 - d)\xi_{\text{crit}} \cos \xi_{\text{crit}} d = 0,$$

$$u_{\text{crit}} = \begin{cases} \sin \xi_{\text{crit}} x, & 0 < x < d, \\ [(1 - x)/(1 - d)] \sin \xi_{\text{crit}} d, & d < x < 1, \end{cases} \qquad (21, 22)$$

where $\xi = \sqrt{k/(v^2-1)}$. The maximum critical speed is $v^2_{max} = 1 + k/\xi^2_{min}$, where ξ_{min} is the minimum positive root of equation (21). v^2_{max} increases with foundation stiffness and decreases with foundation length. The infinite number of critical speeds and their accumulation in the speed region above unity is apparent in Figure 6 which shows the characteristic equation (21) for $k = 20$ and $d = 0{\cdot}6$. Notice the small speed range shown in Figure 6(a). For these values, $v^2_{max} = 2{\cdot}523$.

Positive definiteness of $\tilde{K}$ ensures that an eigenvalue changes stability at each critical speed v^2_{j+1} [12]. Evaluation of the eigenvalue derivatives at the critical speeds yields results analogous to equation (16):

$$\psi = \begin{cases} [v^2_{crit}/(v^2_{crit}-1)](d-x)\sin\xi x, & 0 < x < d, \\ [v^2_{crit}/(v^2_{crit}-1)](d-x)[(1-x)/(1-d)]\sin\xi d, & d < x < 1; \end{cases}$$

$$\mu = \partial(\lambda^2)/\partial(v^2)|_{v^2_{crit}}$$
$$= -(3k/1-d)[(d(1-d) + (\sin\xi d)^2]/[2 + d + (1-d)(\cos\xi d)^2] < 0. \quad (23)$$

The result $\mu < 0$ establishes that one eigenvalue regains stability at each critical speed above unity. This dictates the following stability picture. As the speed increases (quasistatically) past $v^2 = 1$, an infinite number of eigenvalue loci pass from imaginary to real and the trivial equilibrium is divergent unstable. At each of the infinite number of critical speeds above unity, one real eigenvalue passes back to imaginary. All eigenvalues are imaginary above v^2_{max}, and the equilibrium is divergent if, and only if, $1 < v^2 < v^2_{max}$.

Letting $u_1(x)$ and $u_2(x)$ respectively denote the string deflections in the sections with and without elastic foundation, the solution $u_1 = e^{\beta x}$ in equation (5) leads to the dispersion relation

$$(v^2-1)\beta^2 + 2v\lambda\beta + (\lambda^2 + k) = 0$$
$$\beta_{1,2} = [-v\lambda \pm \sqrt{\lambda^2 - k(v^2-1)}]/(v^2-1). \quad (24)$$

The cases $\lambda^2 > k(v^2-1)$, $\lambda^2 < k(v^2-1)$ and $\lambda^2 = k(v^2-1)$ must be treated separately. One can show that real eigenvalues λ exist only for $\lambda^2 < k(v^2-1)$ and $1 < v^2 < v^2_{max}$. In this case, the eigenfunction and characteristic equation are

$$u = \begin{cases} A\, e^{-[\lambda v/(v^2-1)]x} \sin\sqrt{k(v^2-1)-\lambda^2}/(v^2-1), & 0 < x < d, \\ B\, e^{-[\lambda v/(v^2-1)]x}[e^{-[\lambda/(v^2-1)]x} - e^{-2\lambda/(v^2-1)}\, e^{[\lambda/(v^2-1)]x}], & d < x < 1; \end{cases}$$

$$\lambda \tan d\sqrt{k(v^2-1)-\lambda^2}/(v^2-1) + \sqrt{k(v^2-1)-\lambda^2}\tanh\lambda(1-d)/(v^2-1) = 0. \quad (25)$$

Imaginary eigenvalues, including those for $1 < v^2 < v^2_{max}$, are obtained as in reference [4]. All eigenvalue loci are continuous at $v^2 = 1$, in contrast to the discrete spring case. The eigenvalues across a range of speeds are qualitatively similar to the complete foundation eigenvalues discussed below and shown in Figures 7 and 8.

3.4. COMPLETE ELASTIC FOUNDATION

The critical speeds for a complete elastic foundation are found from

$$(v^2 - 1)u_{xx} + ku = 0, \qquad u(0) = u(1) = 0. \tag{26}$$

No non-trivial solutions exist for $v^2 = 1$, and, formally at least, this is not a critical speed. This is consistent with the earlier finding that the stiffness operator is positive definite at this speed. Solution of equation (26) yields

$$v_{\text{crit}}^2 = 1 + k/n^2\pi^2, \qquad u_{\text{crit}} = \sin n\pi x, \qquad n = 1, 2, \ldots . \tag{27}$$

An infinite number of critical speeds exist above $v^2 = 1$, yet they are bounded above by the maximum critical speed $v_{\max}^2 = 1 + k/\pi^2$. Although $v^2 = 1$ is not a critical speed, critical speeds exist arbitrarily close to this speed as $n \to \infty$. The eigenvalue problem (5) is solvable in closed-form as

$$\lambda_n^2 = k(v^2 - 1) - n^2\pi^2(v^2 - 1)^2, \qquad u_n = \mathrm{e}^{-v\lambda_n/[(v^2-1)]x} \sin n\pi x,$$
$$n = 1, 2, \ldots . \tag{28}$$

Eigenvalue derivatives at the critical speeds are

$$\mu_n = \partial(\lambda^2)/\partial(v^2)|_{v_{\text{crit}}^2} = -k. \tag{29}$$

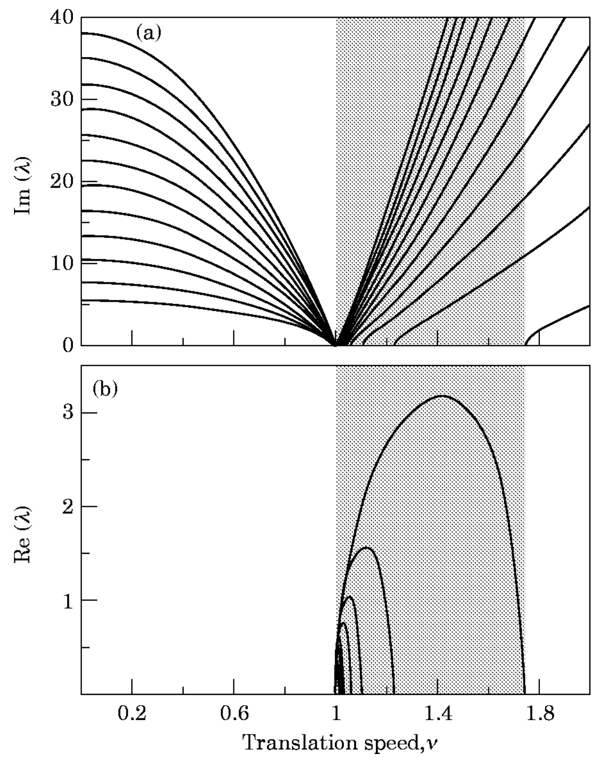

Figure 7. (a) Imaginary and (b) real parts of the eigenvalues for a moving string with complete elastic foundation for $k = 20$. Shaded regions indicate divergence instability; no divergence (or flutter) instability exists for $v < 1$ and $v > 1{\cdot}740$.

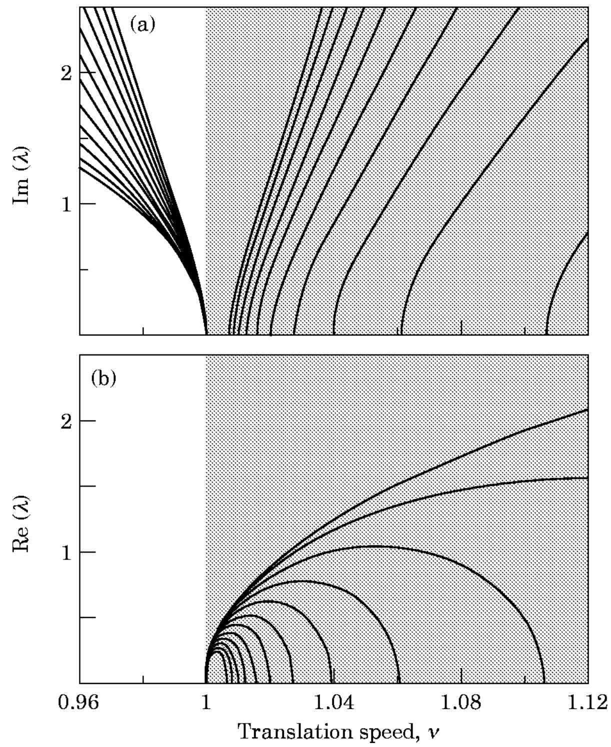

Figure 8. Magnified view of the (a) imaginary and (b) real parts of the eigenvalues near $v = 1$ for the complete elastic foundation case ($k = 20$) of Figure 7. The shaded region indicates divergence instability.

Thus, all eigenvalues are unstable immediately above $v^2 = 1$, and one unstable eigenvalue recovers stability at each critical speed as confirmed by the eigenvalues in Figures 7 and 8 found from equation (28). Each eigenvalue is either real or imaginary but not complex. Thus, flutter instability does not occur, and $v_{max}^2 = 1 + k/\pi^2$ is the maximum unstable speed.

4. SUMMARY AND CONCLUSIONS

The stability of the trivial equilibrium of an elastically supported translating string differs significantly from that of the unsupported translating string and exhibits unique features atypical of gyroscopic continua. These features ae identified analytically from the simple, self-adjoint critical speed eigenvalue problem and a perturbation analysis about the critical speeds. The analytical solution is essential as spatial discretization of the translating string eigenvalue problem can lead to inaccurate, misleading conclusions.

1. Whereas an unsupported string has a single critical speed, an elastic foundation introduces additional critical speeds, all of them greater than the one for an unsupported string. For n discrete spring supports, there are exactly n additional critical speeds. For partial or complete foundations, there are an

infinite number of additional critical speeds. In all cases, there exists a well-defined maximum critical speed v_{max}, a behavior that is unusual among gyroscopic continua where unbounded sequences of critical speeds are typical. Qualitative features of the critical speed spectra are identified by analogies to familiar stationary string vibration problems.

2. Elastically supported strings *always* exhibit divergence instability above the first critical speed. This region of divergence instability extends over $1 < v < v_{max}$ and this is the *only* region of divergence instability. This contrasts with with unsupported strings where the trivial equilibrium is stable for all supercritical speeds. The passage of eigenvalues to and from stability across the critical speeds is determined for each type of elastic foundation using perturbation analysis. The perturbation admits general stability conclusions not available from numerical solution of the translating string eigenvalue problem.

3. For discrete spring supports, the eigenvalues are discontinous across the lowest critical speed. Though the magnitude of the discontinuity can be determined analytically, more focused analysis of this critical speed is required to fully explain this unusual phenomenon.

4. Analytical solutions for gyroscopic continua are rare, and the identified eigenvalue behaviors provide useful benchmarks for evaluating approximate methods applied to gyroscopic problems. The local stability analysis herein also serves as a basis for subsequent study of the non-trivial equilibria.

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